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# Theory and Method of Structural Variations of Finite Element Systems

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**This paper is the continuation of the first author's previous work [Rong, T.-Y., "General Theorems of Topological Variations of Elastic Structures and the Method of Topological Variation," *Acta Mechanica Solida Sinica*, No. 1, 1985, pp. 29–43 (in Chinese)] extending the theory of structural variations presented therein from skeletal structures to the finite element systems in solid continuum. Based on the theory, a new analysis tool, called the structural variation method (SVM), is developed. Its characteristic feature is that it totally eliminates the need for assembling and solving simultaneous equations as required in the usual finite element method. SVM is favorable for the analysis of changing structures and design sensitivities such as that in structural optimization and reliability, elastic-plastic analysis, contact problems, propagation of cracks in solids, etc.**

## I. Introduction

IT is an indispensable procedure for the commonly used finite element method (FEM) to assemble and solve a set of simultaneous equations. However, in many cases, e.g., structural optimization, one has to do that repeatedly, encountering vast amounts of computations. This paper will treat finite element systems from a new point of view and offer an alternative approach to the computational problems, helping overcome the aforementioned difficulties. The main idea of this paper starts with the study of the following three types of elementary structural variations. Type I: change the rigidity of an element and, if necessary, reduce it to zero, leading to the removal of the element from the system.Type II: add a new element to the system. Type III: add a new constraint (or support) to the system, or remove an old one from it. It is apparent that through those three types of elementary variations one can change a system into any other one and can also restore it to the original. The basic theoretical task of this paper is to study how its responses (displacements and stresses) vary and what their explicit formulations are like when a loaded system undergoes those three types of elementary structural variations. The study of those structural variations is synthesized as the theory of structural variations (TSV).

The TSV is established via a fresh concept, called the subelements of an element, which is the downward extension of the usual finite element concept. Through the subelement, one can reveal some interesting properties of finite element systems, which are being stated as five theorems in this paper. These theorems constitute a complete set of explicit formulations sufficient to change a loaded system into any other one and to predict the responses of the varied system. Therefore, the TSV supplies a new analysis tool, called the structural variation method (SVM), totally eliminating the need for assembling and solving simultaneous equations as required in the commonly used FEM. The theory has an application potential in engineering areas and is advantageous to the analysis of changing structures and design sensitivities, such as that in structural optimization and reliability, elastic-plastic analysis, contact problems, propagation of cracks in solids, etc. This theory has been initiated for skeletal structures in Ref. 1, whereas this paper will extend it to the finite element systems in solid continuum. The following sections will discuss them in detail, taking the two-dimensional constant strain triangular element system in linear isotropic elasticity as their representative. However, the final formulations and theorems are also valid for other finite element models in plates, shells, and three-dimensional solids, etc.; Sec.V will give a general procedure for generating subelements from element models in general.

## II. Basic Concepts

Consider a finite element system of $n$ nodes and $m$ triangular elements. Use $\alpha$, $\beta$, ..., to denote the element number and $i, j, m$ its vertices as shown in Fig. 1. The formulations of FEM are well known (see, e.g., Ref. 4):

$$\underline{\varepsilon} = [\varepsilon_x \ \ \varepsilon_y \ \ \gamma_{xy}]^T = \boldsymbol{BD} \tag{1}$$

$$\underline{\sigma} = [\sigma_x \ \ \sigma_y \ \ \tau_{xy}]^T = \boldsymbol{M}\varepsilon \tag{2}$$

$$\boldsymbol{K}^\alpha = At\boldsymbol{B}^T\boldsymbol{MB} \tag{3}$$

$$\boldsymbol{f}^\alpha = \boldsymbol{K}^\alpha\boldsymbol{D} \tag{4}$$

$$\boldsymbol{K} = \sum_{\alpha=1}^{m} \boldsymbol{K}^\alpha \tag{5}$$

$$\boldsymbol{KD} = \boldsymbol{P} \tag{6}$$

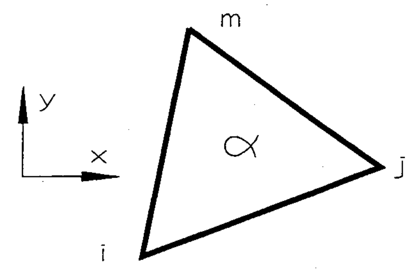


**Fig. 1 Triangular element.**

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$$M = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \frac{E}{(1-\nu^2)} \tag{7}$$

$$B = \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \frac{1}{2A} \tag{8}$$

$$b_i = y_j - y_m, \qquad c_i = -x_j + x_m \tag{9}$$

where $\varepsilon$ is the strain vector; $\sigma$ is the stress vector; $M$ is the elastic matrix; $E$ is the Young's modulus; $\nu$ is Poisson's ratio; $A$ is the area; $f^\alpha$ is the nodal force vector; $K^\alpha$ is the element stiffness matrix; $x_i$ and $y_i$ are the coordinates of the vertices of element $\alpha$; $i, j, m$ are in cyclic permutation; whereas $K$, $D$, and $P$ are the global stiffness matrix, nodal displacement vector, and applied load vector, respectively. The superscript $T$ stands for transpose. The following are some new concepts.

### A. Subelements

Introduce a matrix $Q$ into Eq. (3) such that it makes $Q^T M Q$ diagonal:

$$Q \equiv \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{2}; \qquad Q^{-1} \equiv \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Then Eq. (3) can be rewritten as

$$K^\alpha = H^\alpha W^\alpha (H^\alpha)^T \tag{11}$$

where

$$H^\alpha \equiv A B^T Q^{-T} = \frac{1}{2} \begin{bmatrix} b_i & c_i & b_j & c_j & b_m & c_m \\ b_i & -c_i & b_j & -c_j & b_m & -c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix}^T \tag{12}$$

$$W^\alpha \equiv \mathrm{diag}(W_1^\alpha, W_2^\alpha, W_3^\alpha) \equiv \frac{t}{A} Q^T M Q \tag{13}$$

$$W_1^\alpha \equiv \frac{Et}{2A(1-\nu)}; \quad W_2^\alpha \equiv \frac{Et}{2A(1+\nu)}; \quad W_3^\alpha \equiv \frac{Et}{2A(1+\nu)} \tag{14}$$

Denote each column in $H^\alpha$ by a vector $E_s^\alpha$, $s = 1, 2, 3$:

$$E_1^\alpha \equiv (1/2)[b_i \; c_i \; b_j \; c_j \; b_m \; c_m]^T \tag{15a}$$

$$E_2^\alpha \equiv (1/2)[b_i \; -c_i \; b_j \; -c_j \; b_m \; -c_m]^T \tag{15b}$$

$$E_3^\alpha \equiv (1/2)[c_i \; b_i \; c_j \; b_j \; c_m \; b_m]^T \tag{15c}$$

Thus,

$$H^\alpha = [E_1^\alpha \; E_2^\alpha \; E_3^\alpha] \tag{16}$$

and Eq. (11) can be further rewritten as

$$K^\alpha = \sum_{s=1}^{3} K_s^\alpha \tag{17}$$

where

$$K_s^\alpha \equiv W_s^\alpha E_s^\alpha (E_s^\alpha)^T, \qquad s = 1, 2, 3 \tag{18}$$

Therefore, the matrix $K_s^\alpha$ in Eq. (17) may be regarded as the element stiffness matrix of some subdivided element (having the same vertices as the parent element $\alpha$). It is called the subelement and is denoted by the symbol $\binom{\alpha}{s}$, $s = 1, 2, 3$. The corresponding $W_s^\alpha$ is called the subelement stiffness modulus (or simply modulus), and the vector $E_s^\alpha$ is called the subelement vector of subelement $\binom{\alpha}{s}$. Each triangular element has three subelements.

### B. Generalized Internal Forces, Z-Deformations, and Intrinsic Loads

Introduce three quantities:

$$F^\alpha \equiv [F_1^\alpha \; F_2^\alpha \; F_3^\alpha]^T \equiv t Q \sigma \tag{19}$$

$$Z^\alpha \equiv [Z_1^\alpha \; Z_2^\alpha \; Z_3^\alpha]^T \equiv (H^\alpha)^T D \tag{20}$$

$$P_s^\alpha \equiv W_s^\alpha E_s^\alpha \tag{21}$$

The vector $F^\alpha$ is called the generalized internal force vector (GIF vector), $Z^\alpha$ the generalized deformation vector (or simply Z-deformation vector) of element $\alpha$, and $P_s^\alpha$ the intrinsic load vector of subelement $\binom{\alpha}{s}$, $s = 1, 2, 3$. Please notice the following notational convention: When matrices (or vectors) of different dimensions appear together in an operation, the matrix (or vector) of lower dimension is supposed to have been extended to a matrix of the same dimension as the higher one by inserting zero entries in appropriate locations. For instance, the matrix $(K^\alpha)_{6\times 6}$ in

$$(K)_{2n\times 2n} = \sum_{\alpha=1}^{m} K^\alpha$$

should be considered to have been extended to a matrix $(K^\alpha)_{2n\times 2n}$ with some zero entries inserted in the positions where it has no contributions to $K$. So is the matrix $(H^\alpha)_{6\times 3}$ in $Z^\alpha \equiv (H^\alpha)^T D$, where $D$ is of $2n \times 2n$, and other matrices or vectors throughout the paper where needed.

From Eqs. (19), (2), (1), (12), (13), and (20), one has

$$F^\alpha = t Q^T M B D = W^\alpha (H^\alpha)^T D = W^\alpha Z^\alpha \tag{22}$$

Therefore, $W^\alpha$ is the coefficient matrix between the GIF vector $F^\alpha$ and the Z-deformation vector $Z^\alpha$ of element $\alpha$. More generally, collecting all $F^\alpha$, $Z^\alpha$, and $W^\alpha$, $\alpha = 1, 2, \ldots, m$, to make their global counterparts, denoted by $(F)_{3m\times 1}$, $(Z)_{3m\times 1}$, and $(W)_{3m\times 3m}$ (diagonal), respectively, one has the global relationship as

$$F = WZ \tag{23}$$

With $F^\alpha$ and $Z^\alpha$ known, from Eqs. (19), (1), and (12), the stress vector $\sigma$ and strain vector $\varepsilon$ are calculated by

$$\sigma = Q^{-1} F^\alpha / t, \qquad \varepsilon = Q Z^\alpha / A \tag{24}$$

### C. Basic Displacements and Basic Internal Forces

Put the six components of the intrinsic load vector $P_s^\alpha$ of subelement $\binom{\alpha}{s}$ on the corresponding nodal degrees of freedom (DOFs) of element $\alpha$; then the system will deform. The global displacement vector produced by it is denoted by $V_s^\alpha$ and defined as

$$V_s^\alpha \equiv K^{-1} P_s^\alpha \tag{25}$$

which is called the basic displacement vector (BD vector) of subelement $\binom{\alpha}{s}$. The Z-deformation of subelement $\binom{\beta}{r}$, formed from BD vector $V_s^\alpha$ of subelement $\binom{\alpha}{s}$, is particularly denoted by the symbol $Z_{rs}^{\beta\alpha}$:

$$Z_{rs}^{\beta\alpha} \equiv (E_r^\beta)^T V_s^\alpha, \qquad \alpha, \beta = 1, 2, \ldots, m; \qquad r, s = 1, 2, 3 \tag{26}$$

If $\binom{\alpha}{s} = \binom{\beta}{r}$, then $Z_{ss}^{\alpha\alpha}$ is called the principal Z-deformation. Use the symbol $\binom{\ell}{r}$, $r = 1, 2$, to denote a DOF ($r = 1$ for $x$ direction and $r = 2$ for $y$ direction, respectively) of the node $\ell$. A unit load vector is symbolized by $\bar{P}_r^\ell$, indicating its unique nonzero component $\bar{P}_r^\ell =$

1 at $\binom{\ell}{r}$, but zero elsewhere. The GIF vector $\boldsymbol{F}^{\alpha}$ produced by a unit load $\overline{\boldsymbol{P}}_r^{\ell}$ is particularly called the basic internal force vector (BIF vector) of element α and denoted by $\overline{\boldsymbol{F}}_{\bullet r}^{\alpha\ell} \equiv [\overline{F}_{1r}^{\alpha\ell}, \overline{F}_{2r}^{\alpha\ell}, \overline{F}_{3r}^{\alpha\ell}]^T$. If $\overline{\boldsymbol{F}}_{\bullet r}^{\alpha\ell}$ is known for all of the DOFs, then the GIF vector $\boldsymbol{F}^{\alpha}$ produced by any external load vector $\boldsymbol{P} = [P_1^1, P_2^1, \dots, P_2^n]^T$ is calculated by

$$\boldsymbol{F}^{\alpha} = \sum_{\ell=1}^{n} \sum_{r=1}^{2} \overline{\boldsymbol{F}}_{\bullet r}^{\alpha\ell} P_r^{\ell} \tag{27}$$

Note that the symbol $\binom{\ell}{r}$ for a DOF is distinct from the symbol $\binom{\alpha}{s}$ for a subelement in the superscript in Greek.

## III. General Properties of Finite Element Systems in Connection with Basic Displacements

With the preceding introduced concepts, one can bring to light some interesting properties of finite element systems, which are stated as five theorems. They constitute a complete set of explicit formulations sufficient to carry out the elementary structural variations of types I, II, and III. Since these theorems have been symbolically proven in detail in Ref. 1 for skeletal structures and are also valid for the finite element systems in general, they are simply listed next to avoid repetition. Nevertheless, short proofs of them are given in the Appendix, and clarifications will be made where the necessity arises to adapt them to the finite element systems in general.

### A. General Identities

*Theorem 1* (Reciprocal Theorem of Basic Displacements and Basic Internal Forces): In a finite element system, the component $V_{sr}^{\alpha\ell}$ of the BD vector $\boldsymbol{V}_s^{\alpha}$ at DOF $\binom{\ell}{r}$ is identical to the component $\overline{F}_{sr}^{\alpha\ell}$ of the BIF vector $\overline{\boldsymbol{F}}_{\bullet r}^{\alpha\ell}$ of element α, i.e.,

$$V_{sr}^{\alpha\ell} = \overline{F}_{sr}^{\alpha\ell} \tag{28}$$

or

$$\boldsymbol{V}_{\bullet r}^{\alpha\ell} = \overline{\boldsymbol{F}}_{\bullet r}^{\alpha\ell} \tag{29}$$

where $(\boldsymbol{V}_{\bullet r}^{\alpha\ell})_{3\times 1} \equiv [V_{1r}^{\alpha\ell}, V_{2r}^{\alpha\ell}, V_{3r}^{\alpha\ell}]^T$ is a vector of the three components at $\binom{\ell}{r}$, each from one BD vector $\boldsymbol{V}_s^{\alpha}$, $s = 1, 2, 3$. Theorem 1 indicates that $\boldsymbol{V}_s^{\alpha}$ is actually the influence coefficient vector of the generalized internal force $F_s^{\alpha}$. Therefore, the generalized internal force produced by any external load $\boldsymbol{P}$ may be calculated by

$$F_s^{\alpha} = (\boldsymbol{V}_s^{\alpha})^T \boldsymbol{P} \quad \text{or} \quad \boldsymbol{F}^{\alpha} = \boldsymbol{V}^{\alpha} \boldsymbol{P} \tag{30}$$

where $(\boldsymbol{V}^{\alpha})_{3\times 2n}$ is the matrix of the three rows $(\boldsymbol{V}_s^{\alpha})^T$, $s = 1, 2, 3$. Collecting all $\boldsymbol{V}^{\alpha}$, $\alpha = 1, 2, \dots, m$, to make the global basic displacement matrix $(\boldsymbol{V})_{3m\times 2n}$, one has

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{P} \tag{31}$$

*Theorem 2* (Explicit Decomposition Theorem on the Inverse of the Global Stiffness Matrix): The inverse of the global stiffness matrix $\boldsymbol{K}$ of a finite element system can be expressed explicitly in terms of the global BD matrix $\boldsymbol{V}$ and the global diagonal stiffness modulus matrix $\boldsymbol{W}$, i.e.,

$$\boldsymbol{K}^{-1} = \boldsymbol{V}^T \boldsymbol{W}^{-1} \boldsymbol{V} \tag{32}$$

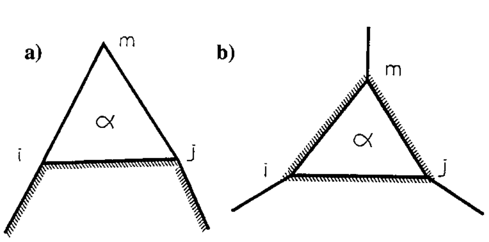


**Fig. 2 a) Branching element; b) connecting element.**

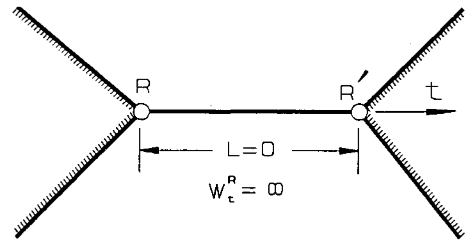


**Fig. 3 Constraint-subelement.**

Therefore, the displacement vector $\boldsymbol{D}$ produced by any external load $\boldsymbol{P}$ may be calculated by using any of the following four formulas:

$$\boldsymbol{D} = \boldsymbol{K}^{-1}\boldsymbol{P} = \boldsymbol{V}^T\boldsymbol{W}^{-1}\boldsymbol{V}\boldsymbol{P} = \boldsymbol{V}^T\boldsymbol{W}^{-1}\boldsymbol{F} = \boldsymbol{V}^T\boldsymbol{Z} \tag{33}$$

*Theorem 3* (Reciprocal Substitution Theorem of Z-Deformations): In a finite element system, any pair of Z-deformations formed from the basic displacements of any two subelements can be substituted one for another via their stiffness moduli, i.e.,

$$W_s^{\alpha} Z_{sr}^{\alpha\beta} = W_r^{\beta} Z_{rs}^{\beta\alpha} \quad \text{or} \quad Z_{rs}^{\beta\alpha} = Z_{sr}^{\alpha\beta} W_s^{\alpha} / W_r^{\beta} \tag{34}$$

Theorems 1 and 2 indicate that $\boldsymbol{F}$ [or σ with Eq. (24)] and $\boldsymbol{D}$ produced by any external load $\boldsymbol{P}$ may be calculated straightforwardly via $\boldsymbol{V}$, whereas $\boldsymbol{V}$ is independent of external loads but determined by the system itself only. Therefore, to obtain the responses of a loaded system undergoing the structural variations of types I, II, and III, it is sufficient to get the varied $\boldsymbol{V}$, which is being discussed later.

### B. Elementary Structural Variations of Type I

*Theorem 4* (Theorem on the Structural Variations of Type I): The varied basic displacements of a finite element system after the variation of a subelement $\binom{\alpha}{s}$ in its stiffness modulus, $\hat{W}_s^{\alpha} = W_s^{\alpha} + \Delta W_s^{\alpha}$, are given by

$$\hat{\boldsymbol{V}}_s^{\alpha} = \boldsymbol{V}_s^{\alpha}(1 + m_s^{\alpha})/(1 + m_s^{\alpha} Z_{ss}^{\alpha\alpha}) \tag{35}$$

$$\hat{\boldsymbol{V}}_r^{\beta} = \boldsymbol{V}_r^{\beta} - \boldsymbol{V}_s^{\alpha} Z_{sr}^{\alpha\beta} m_s^{\alpha}/(1 + m_s^{\alpha} Z_{ss}^{\alpha\alpha}), \qquad \binom{\beta}{r} \neq \binom{\alpha}{s} \tag{36}$$

where $m_s^{\alpha} \equiv \Delta W_s^{\alpha}/W_s^{\alpha}$ is the variation factor of $\binom{\alpha}{s}$; $\hat{\boldsymbol{V}}_s^{\alpha}$ stands for the varied BD vector of $\binom{\alpha}{s}$, and hereafter all of the varied quantities will be denoted by the original symbol with ( $\hat{}$ ). It is seen from Eq. (14) that every parameter contained in $W_s^{\alpha}$, e.g., $t$, $E$, or ν, may cause all of the three subelements to vary. However, it is sufficient to use Theorem 4 three times, one after another in turn, $s$ =1, 2, 3, to complete the variations of type I. If setting $m_s^{\alpha} = -1$ in Eq. (36), the subelement $\binom{\alpha}{s}$ will be removed. Using this for $s$ = 1, 2, 3 will result in removing the element α.

### C. Elementary Structural Variations of Type II

The type II structural variations involve two cases. Case 1: a new element, say α, branches out from two original nodes $i$ and $j$, and a new node $m$ occurs at the same time; the element added in this way is called the branching element as shown in Fig. 2a. Case 2: a new element α is added, connecting three existing nodes $i$, $j$, and $m$, without any new node occurring as shown in Fig. 2b; it is called the connecting element. The two cases are discussed separately later to find the varied BD matrix $\hat{\boldsymbol{V}}$ after adding an element α.

To add a branching element to the system, it is necessary to introduce the concept of constraint-subelement. It is a special case of the beam subelement defined in Ref. 1 for skeletal structures. It has the following features. 1) A constraint-subelement, denoted by $\binom{R}{t}$, can merge two nodes $R$ and $R'$ into one in its axial direction $t$ as shown in Fig. 3. 2) Its length $L = 0$, whereas its stiffness modulus, denoted by $W_t^R$, $W_t^R = \infty$. 3) Its subelement vector, denoted by $\boldsymbol{E}_t^R$, is

$$\boldsymbol{E}_t^R = [-1, 1]^T \tag{37}$$

where the values −1 and 1 correspond to the two DOFs of the nodes $R$ and $R'$ in its axial direction $t$, respectively. With this concept, the local structure of a hinge joint, say $j$, can be regarded as a pair of constraint subelements $\binom{R}{t}$, $t = 1$ ($x$ direction) and $t = 2$ ($y$ direction), as shown in Fig. 4a; the nodes $R$ and $j$ are actually located at the same point. Thus, a branching element can be treated as the combination of a simply supported element α as shown in Fig. 4b or Fig. 4c and a constraint subelement $\binom{R}{t}$ between $R$ and $j$ with $t = 1$ for Fig. 4b or $t = 2$ for Fig. 4c. Therefore, adding a branching element to the system can be carried out through two steps. First add a simply supported element α and then a constraint subelement $\binom{R}{t}$.

### *1. Adding a Simply Supported Element*

To add a simply supported element to a system, one should notice two facts that every intrinsic load vector $\boldsymbol{P}_s^\alpha = \boldsymbol{W}_s^\alpha \boldsymbol{E}_s^\alpha$ is a self-equilibriated load set, which may be verified directly from the definition (21), and that any self-equilibriated load set applied to the DOFs of a simply supported element produces no displacements at any original DOFs. Accordingly, the six DOFs of the simply supported element α are divided into two groups: group A includes the original DOFs $\binom{i}{1}$, $\binom{j}{2}$, and $\binom{j}{2}$, and group B includes $\binom{R}{1}$, $\binom{m}{1}$, and $\binom{m}{2}$ (for the case of Fig. 4b), which are the new DOFs to the original system. Based on the aforementioned facts and the definition (25), it is sufficient to find out the components of the varied $\hat{\boldsymbol{V}}^\alpha$ at DOFs of group B. For this, first express the nodal force vector $\boldsymbol{f}^\alpha$ in terms of $\boldsymbol{F}^\alpha$ by using Eqs. (4), (11), and (22):

$$\boldsymbol{f}^\alpha = \boldsymbol{H}^\alpha \boldsymbol{W}^\alpha (\boldsymbol{H}^\alpha)^T \boldsymbol{D} = \boldsymbol{H}^\alpha \boldsymbol{F}^\alpha \tag{38}$$

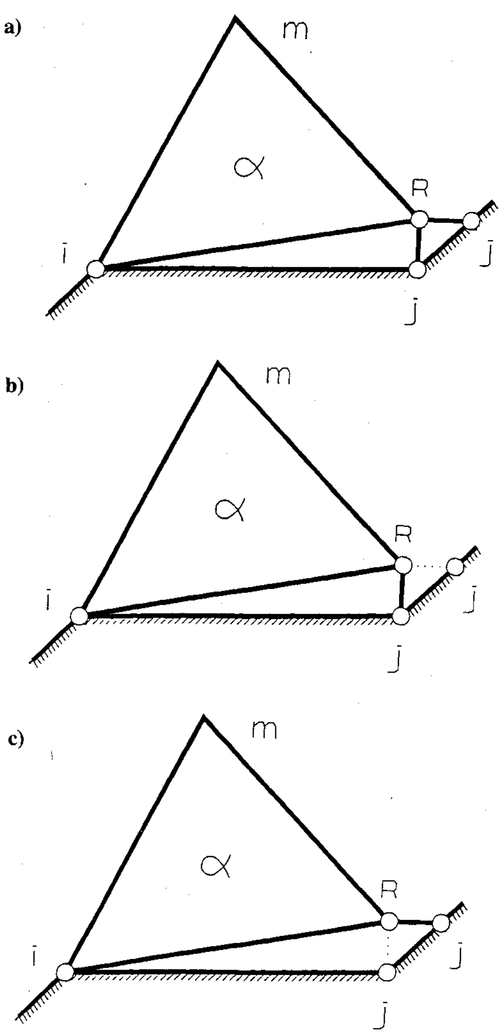


**Fig. 4 a) Pair of constraint-subelements acting as a hinge joint; b) and c) simply supported elements.**

or in the partition form,

$$\boldsymbol{f}^\alpha = \begin{bmatrix} \boldsymbol{f}_A^\alpha \\ \boldsymbol{f}_B^\alpha \end{bmatrix} = \boldsymbol{H}^\alpha \boldsymbol{F}^\alpha = \begin{bmatrix} \boldsymbol{H}_A^\alpha \\ \boldsymbol{H}_B^\alpha \end{bmatrix} \boldsymbol{F}^\alpha \tag{39}$$

where $\boldsymbol{f}_A^\alpha$ and $\boldsymbol{H}_A^\alpha$ are the partitions of $\boldsymbol{f}^\alpha$ and $\boldsymbol{H}^\alpha$, respectively, corresponding to group A; whereas the partitions $\boldsymbol{f}_B^\alpha$ and $\boldsymbol{H}_B^\alpha$ correspond to group B. From Eq. (12) one has

$$\boldsymbol{H}_A^\alpha = \frac{1}{2}\begin{bmatrix} b_i & b_i & c_i \\ c_i & -c_i & b_i \\ c_j & -c_j & b_j \end{bmatrix}, \qquad \boldsymbol{H}_B^\alpha = \frac{1}{2}\begin{bmatrix} b_j & b_j & c_j \\ b_m & b_m & c_m \\ c_m & -c_m & b_m \end{bmatrix} \tag{40}$$

Thus, from Eq. (39) one has

$$\boldsymbol{f}_B^\alpha = \boldsymbol{H}_B^\alpha \boldsymbol{F}^\alpha \tag{41}$$

Substituting the unit load $\bar{\boldsymbol{P}}_r^\ell$ at each DOF of group B for $\boldsymbol{f}_B^\alpha$ in Eq. (41) and noting the definition of $\bar{\boldsymbol{F}}_{\bullet r}^{\alpha\ell}$ yield

$$\bar{\boldsymbol{P}}_r^\ell = \boldsymbol{H}_B^\alpha \bar{\boldsymbol{F}}_{\bullet r}^{\alpha\ell}, \qquad \binom{\ell}{r} \in B \tag{42}$$

or in the matrix form for the three DOFs $\binom{\ell}{r} = \binom{R}{1}$, $\binom{m}{1}$, and $\binom{m}{2}$, one has

$$\boldsymbol{I} = \boldsymbol{H}_B^\alpha \bar{\boldsymbol{F}}_{\bullet\bullet}^{\alpha B} \tag{43}$$

where $\boldsymbol{I}_{3\times 3}$ is a unit matrix and $(\bar{\boldsymbol{F}}_{\bullet\bullet}^{\alpha B})_{3\times 3}$ stands for the matrix of the three BIF vectors of $\bar{\boldsymbol{F}}_{\bullet r}^{\alpha\ell}$ produced by the three $\bar{\boldsymbol{P}}_r^\ell$ applied at $\binom{\ell}{r} = \binom{R}{1}$, $\binom{m}{1}$, and $\binom{m}{2}$ individually. According to Theorem 1, the desired BD components at the DOFs of group B of the element α, denoted by $(\hat{\boldsymbol{V}}_{\bullet\bullet}^{\alpha B})_{3\times 3}$, can be expressed from Eq. (43) as $(\boldsymbol{H}_B^\alpha)^{-1}$, i.e.,

$$\hat{\boldsymbol{V}}_{\bullet\bullet}^{\alpha B} = \frac{1}{2c_m A}\begin{bmatrix} (b_m^2 + c_m^2), & -(b_j b_m + c_j c_m), & 2A \\ (c_m^2 - b_m^2), & (b_j b_m - c_j c_m), & -2A \\ -2b_m c_m, & 2b_j c_m, & 0 \end{bmatrix} \tag{44}$$

Similarly, for the case of Fig. 4c, group B consists of $\binom{R}{2}$, $\binom{m}{1}$, and $\binom{m}{2}$, and

$$\hat{\boldsymbol{V}}_{\bullet\bullet}^{\alpha B} = \frac{1}{2b_m A}\begin{bmatrix} -(b_m^2 + c_m^2), & 2A, & (b_j b_m + c_j c_m) \\ (b_m^2 - c_m^2), & 2A, & (c_j c_m - b_j b_m) \\ 2b_m c_m, & 0, & -2b_m c_j \end{bmatrix} \tag{45}$$

Next, consider the new BD matrix $\hat{\boldsymbol{V}}^\beta$ of any original element β after adding the simply supported element α. For this, one should note another fact that the displacements produced by any load applied at the original DOFs of the system remain unchanged except for the new components at the DOFs of group B, and the simply supported element α itself has no deformations, i.e., $\boldsymbol{Z}^\alpha = (\boldsymbol{H}^\alpha)^T \boldsymbol{D} = \boldsymbol{0}$ or $(\boldsymbol{H}_A^\alpha)^T \boldsymbol{D}_A + (\boldsymbol{H}_B^\alpha)^T \boldsymbol{D}_B = \boldsymbol{0}$, where $\boldsymbol{D}_A$ and $\boldsymbol{D}_B$ are the displacement components at DOFs of group A and B, respectively. Thus, one has

$$\boldsymbol{D}_B = -(\boldsymbol{H}_B^\alpha)^{-T} (\boldsymbol{H}_A^\alpha)^T \boldsymbol{D}_A = \Omega\, \boldsymbol{D}_A \tag{46}$$

where for the case of Fig. 4b

$$\Omega \equiv -(\boldsymbol{H}_B^\alpha)^{-T} (\boldsymbol{H}_A^\alpha)^T = \frac{1}{c_m}\begin{bmatrix} c_m & -b_m & b_m \\ c_m & b_j & -b_j \\ 0 & -c_i & -c_j \end{bmatrix} \tag{47}$$

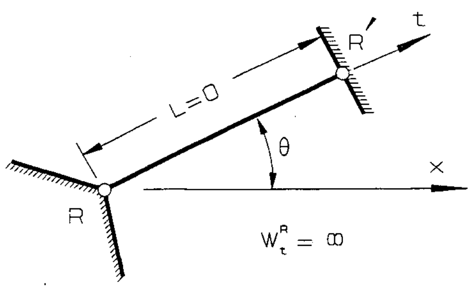


**Fig. 5 Support-subelement.**

or

$$(D_B)^T = (D_A)^T\Omega^T \tag{48}$$

The definition (25) indicates that the BD vector $\hat{V}_r^\beta$ of any original subelement $\binom{\beta}{r}$ is just the case mentioned earlier. Therefore, the components of $\hat{V}_r^\beta$ remain the same at the original DOFs as before, and the new components of $\hat{V}_r^\beta$ at the DOFs of group B, denoted by $(\hat{V}_{r\bullet}^{\beta B})_{1\times 3}$, can be obtained by using Eq. (48):

$$\hat{V}_{r\bullet}^{\beta B} = V_{r\bullet}^{\beta A}\Omega^T, \qquad \beta \neq \alpha, \qquad r = 1, 2, 3 \tag{49}$$

where $(V_{r\bullet}^{\beta A})_{1\times 3}$ is the row vector of the original components of $V_r^\beta$ at DOFs of group A. Collecting the three row vector equations (49) for element β, one has

$$\hat{V}_{\bullet\bullet}^{\beta B} = \hat{V}_{\bullet\bullet}^{\beta A}\Omega^T, \qquad \beta \neq \alpha \tag{50}$$

where $\hat{V}_{\bullet\bullet}^{\beta B}$ and $V_{\bullet\bullet}^{\beta A}$ are the matrices of the three row vectors $\hat{V}_{r\bullet}^{\beta B}$ and $\hat{V}_{r\bullet}^{\beta A}$, $r$ = 1, 2, 3, respectively. For the case of Fig. 4c, the matrix Ω is evaluated by

$$\Omega = \frac{1}{b_m}\begin{bmatrix} -c_m & b_m & c_m \\ -b_i & 0 & -b_j \\ c_j & b_m & -c_j \end{bmatrix} \tag{51}$$

### 2. *Inserting a Constraint-Subelement*

After the first step, a constraint-subelement $\binom{R}{t}$ is to be inserted between $R$ and $j$ in the $x$ direction (Fig. 4b) or in the $y$ direction (Fig. 4c). This is just the same procedure as that of adding a support-subelement that has been discussed in detail to obtain Eq. (4-26) in Ref. 1. Next is listed the final result to avoid repetition.

A constraint-subelement $\binom{R}{t}$ inserted into the system will cause the BD vector $V_r^\beta$ of any subelement $\binom{\beta}{r}$ of the system to become

$$\hat{V}_r^\beta = V_r^\beta - \mathring{V}_T^R Z_{tr}^{R\beta}/\mathring{Z}_{tt}^{RR}, \qquad \beta = 1, 2, \ldots, m; \qquad r = 1, 2, 3 \tag{52}$$

where $\mathring{V}_t^R \equiv K^{-1}E_t^R$ is the auxiliary BD vector of $\binom{R}{t}$, which can be readily obtained from Eq. (33), whereas $\mathring{Z}_{tr}^{R\beta} = (E_t^R)^T V_r^\beta$ and $\mathring{Z}_{tt}^{RR} = (E_t^R)^T \mathring{V}_t^R$. Therefore, by using Eqs. (44), (45), (50), and (52), one can accomplish the addition of a branching element to the system.

### 3. *Adding a Connecting Element*

When a connecting element α is added to the system, no new node occurs. Therefore, it is sufficient to deal with its subelement $\binom{\alpha}{s}$, $s$ =1, 2, 3. Nevertheless, this is just the case of adding a connecting beam-subelement to a system, which has been discussed in detail, resulting in Eqs. (4-23) and (4-24) of Ref. 1, and which is also valid for the present case. Next is given the conclusion from Ref. 1.

When a connecting subelement $\binom{\alpha}{s}$ is added to the system, the BD vectors become

$$\hat{V}_s^\alpha = \mathring{V}_s^\alpha/(1 + \mathring{Z}_{ss}^{\alpha\alpha}) \tag{53}$$

$$\hat{V}_r^\beta = V_r^\beta - \mathring{V}_s^\alpha Z_{sr}^{\alpha\beta}/(1 + \mathring{Z}_{ss}^{\alpha\alpha}), \qquad \binom{\beta}{r} \neq \binom{\alpha}{s} \tag{54}$$

where $\mathring{V}_s^\alpha = K^{-1}P_s^\alpha$ is the auxiliary BD vector of $\binom{\alpha}{s}$, which may be calculated readily from Eq. (33), and $P_s^\alpha$ is the intrinsic load vector of $\binom{\alpha}{s}$ to be added, whereas $Z_{sr}^{\alpha\beta} = (E_s^\alpha)^T V_r^\beta$ and $\mathring{Z}_{ss}^{\alpha\alpha} = (E_s^\alpha)^T \mathring{V}_s^\alpha$. Summarizing all that has been discussed in this subsection yields the conclusion of the structural variations of type II, stated as a theorem:

*Theorem 5* (Theorem on the Structural Variations of Type II): When a simply supported element is added to a system, the original basic displacements remain unchanged except for their new components [Eq. (50)] at the new DOFs, whereas those of the added element itself have nonzero components [Eq. (44) or (45)] at the new DOFs only; if a constraint subelement or a connecting subelement is inserted among the original nodes, the BD vectors are determined and/or modified by its auxiliary BD vectors [Eqs. (52–54)].

## D. Elementary Structural Variations of Type III

The type III structural variations also have two cases to be discussed. One is to insert a support-subelement defined in Ref. 1, symbolized by $\binom{R}{t}$, between a node $R$ of the system and the ground (denoted by $R'$) along its axial direction $t$, as shown in Fig. 5. Nevertheless, if the point $R'$ is treated as a node of the system, then the support-subelement $\binom{R}{t}$ is just a constraint-subelement. So Eq. (52) is equally useful for adding a support-subelement. The second case is to remove an existing support-subelement $\binom{R}{t}$ from the system. Upon removing a support-subelement $\binom{R}{t}$ from the system, the basic displacements of a subelement $\binom{\alpha}{s}$ will become

$$\hat{V}_s^\alpha = V_s^\alpha + V_t^R\eta_{ts}^{R\alpha} \tag{55}$$

where

$$\eta_{ts}^{R\alpha} \equiv W_s^\alpha Z_{st}^{\alpha R} \Big/ \left[\sum_{\beta=1}^{q} (T_t^\beta)^T W^\beta Z_{\bullet t}^{\beta R}\right] \tag{56}$$

$$T_t^\beta \equiv -(H_R^\beta)^T R_t^R \tag{57}$$

$$R_t^R \equiv [\cos\theta, \sin\theta]^T \tag{58}$$

where $V_s^\alpha$ and $V_t^R$ are the original BD vectors of $\binom{\alpha}{s}$ and $\binom{R}{t}$, respectively; $q$ is the total number of the elements around the support node $R$; θ is the angle between $\binom{R}{t}$ and the $x$ axis (Fig. 5); $(H_R^\beta)_{2\times 3}$ is the partition of $H^\beta$ corresponding to the node $R$, and $Z_{\bullet t}^{\beta R} = (H^\beta)^T V_t^R$ is the Z-deformation vector of element β from $V_t^R$. Equation (55) is equivalent to Eq. (4-31) in Ref. 1, adapted to the finite element system in general.

# IV. Structural Variation Method for Structural Analysis

Based on the concepts and theorems introduced in the preceding sections, a fresh way can be created for structural analysis. The new method may be described as follows. Select an element arbitrarily from the system and fix it at some three DOFs of it on the ground, treating it as the initial structure (a simply supported element), from which will grow out other elements. The basic displacements of the initial structure is known, i.e., Eq. (44) or (45). Then add the elements connected to it and the constraint subelements needed one after another until all of the elements of the system are completed by using Theorem 5. Then (or in the meantime) insert the supports needed by using Eq. (52) or delete the excess constraints over the real boundary condition by using Eq. (55). Thus, the desired system and its $V$ have been obtained at the same time. With $V$, one can calculate the σ, ε, and $D$ produced by any $P$ readily from Eqs. (31), (24), and (33). If any structural variations are needed to modify the structure, one can also use Theorems 1–5 to modify $V$, then obtain the new responses, until satisfaction is met. Since the analysis herein is carried out by means of the structural variations, totally eliminating the need for assembling and solving simultaneous equations, it is called the structural variation method (SVM).

### A. Example

The system (plane stress) shown in Fig. 6c has $E = 1.0$, $\nu = 0.3$, and $t = 1.0$; find the stresses $\sigma$ and the displacements $\boldsymbol{D}$ produced by the load $\boldsymbol{P}$ shown in the figure. From Eqs. (9), (13), and (14), one has the initial data: for element 1, $b_1 = -1$, $b_2 = 1$, $b_3 = 0$, $c_1 = -1$, $c_2 = 0$, $c_3 = 1$, and $A = 0.5$; for element 2, $b_3 = -1$, $b_2 = 0$, $b_4 = 1$, $c_3 = 0$, $c_2 = -1$, $c_4 = 1$, and $A = 0.5$; $\boldsymbol{W}^1 = \boldsymbol{W}^2 = \mathrm{diag}(10/7, 10/13, 10/13)$. The solution procedure is as follows.

*Step 1*

Take element 1 as the initial structure (Fig. 6a). The DOFs of group B are $\binom{2}{1}$, $\binom{3}{1}$, and $\binom{3}{2}$. Substituting the related data of element 1 into Eq. (44) yields

$$\boldsymbol{V}^1 = \begin{bmatrix} (\boldsymbol{V}_1^1)^T \\ (\boldsymbol{V}_2^1)^T \\ (\boldsymbol{V}_3^1)^T \end{bmatrix} = \overset{\text{node 2} \quad \text{node 3}}{\begin{bmatrix} 1, 0 & \vdots & 0, & 1 \\ 1, 0 & \vdots & 0, & -1 \\ 0, 0 & \vdots & 2, & 0 \end{bmatrix}}$$

where the zero components at node 1 are ignored and also are in the following steps.

*Step 2*

Add the simply supported element 2 to the initial structure shown in Fig. 6b, where the double circle at node $R$ indicates that node $R$ is free in the $x$ direction at the moment as has been illustrated in Fig. 4b. The DOFs of group A are $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{2}{2}$, whereas those of group B are $\binom{R}{1}$, $\binom{4}{1}$, and $\binom{4}{2}$. From Eqs. (47), (49), and (44), one has

$$\Omega = \frac{1}{c_4}\begin{bmatrix} c_4 & -b_4 & b_4 \\ c_4 & b_2 & -b_2 \\ 0 & -c_3 & -c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{V}^{1B}_{\bullet\bullet} = \boldsymbol{V}^{1A}_{\bullet\bullet}\Omega^T = \underset{\text{node 3} \;\; \binom{2}{2}}{\begin{bmatrix} 0 & 1 & \vdots & 0 \\ 0 & -1 & \vdots & 0 \\ 2 & 0 & \vdots & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \underset{\binom{R}{1} \;\; \text{node 4}}{\begin{bmatrix} -1 & \vdots & 0 & 0 \\ 1 & \vdots & 0 & 0 \\ 2 & \vdots & 2 & 0 \end{bmatrix}}$$

$$\boldsymbol{V}^{2B}_{\bullet\bullet} = \underset{\binom{R}{1} \;\; \text{node 4}}{\begin{bmatrix} 2 & \vdots & 1 & 1 \\ 0 & \vdots & 1 & -1 \\ -2 & \vdots & 0 & 0 \end{bmatrix}}$$

Thus, the BD matrix $\boldsymbol{V}$ of the system in Fig. 6b is

$$\boldsymbol{V} = \begin{bmatrix} V^1 \\ V^2 \end{bmatrix} = \underset{\binom{R}{1} \;\; \text{node 2} \;\; \text{node 3} \;\; \text{node 4}}{\begin{bmatrix} -1 & \vdots & 1, 0 & \vdots & 0, & 1 & \vdots & 0, & 0 \\ 1 & \vdots & 1, 0 & \vdots & 0, & -1 & \vdots & 0, & 0 \\ 2 & \vdots & 0, 0 & \vdots & 2, & 0 & \vdots & 2, & 0 \\ 2 & \vdots & 0, 0 & \vdots & 0, & 0 & \vdots & 1, & 1 \\ 0 & \vdots & 0, 0 & \vdots & 0, & 0 & \vdots & 1, & -1 \\ -2 & \vdots & 0, 0 & \vdots & 0, & 0 & \vdots & 0, & 0 \end{bmatrix}}$$

*Step 3*

Insert the constraint subelement $\binom{R}{1}$ between the nodes $R$ and 2 in the $x$ direction. For this, apply a pair of unit forces, i.e., $\boldsymbol{E}_1^R = [-1, 1]^T$ at $\binom{R}{1}$ and $\binom{2}{1}$ (Fig. 6b), to calculate its auxiliary BD vector $\mathring{\boldsymbol{V}}_1^R$ by using Eq. (33)

$$\mathring{\boldsymbol{V}}_1^R = \boldsymbol{V}^T\boldsymbol{W}^{-1}\boldsymbol{V}\boldsymbol{E}_1^R = [-14.16 \;\vdots\; 1.4, 0.0 \;\vdots\; -5.2, 1.4 \;\vdots\; -6.6, -1.4]^T$$

$$\mathring{Z}_{11}^{RR} = (\boldsymbol{E}_1^R)^T\, \mathring{\boldsymbol{V}}_1^R = (-1) \times (-14.6) + 1 \times (1.4) = 16$$

and use Eq. (26) to calculate $Z_{1r}^{R\beta}$ from $\boldsymbol{V}_r^\beta$ for every $\binom{\beta}{r}$ needed in Eq. (52); they are [2, 0, –2, –2, 0, 2]. Therefore, evaluating Eq. (52) yields the final BD vectors of the desired system (Fig. 6c):

$$\boldsymbol{V}_{\text{final}} = \underset{\text{node 2} \qquad\qquad \text{node 3} \qquad\qquad \text{node 4}}{\begin{bmatrix} 0.825, & 0 & \vdots & 0.65, & 0.825 & \vdots & 0.825, & 0.175 \\ 1.0, & 0 & \vdots & 0, & -1.0 & \vdots & 0, & 0 \\ 0.175, & 0 & \vdots & 1.35, & 0.175 & \vdots & 1.175, & -0.175 \\ 0.175, & 0 & \vdots & -0.65, & 0.175 & \vdots & 0.175, & 0.825 \\ 0, & 0 & \vdots & 0, & 0 & \vdots & 1.0, & -1.0 \\ -0.175, & 0 & \vdots & 0.65, & -0.175 & \vdots & 0.825, & 0.175 \end{bmatrix}}$$

*Step 4*

Load the system with $\boldsymbol{P} = [0.5, 0.5]^T$ applied at the $\binom{3}{2}$ and $\binom{4}{2}$ to obtain $\underline{\sigma}$ and $\boldsymbol{D}$. From Eqs. (23), (24), and (33) one has

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{P} = [\overset{\text{element 1}}{0.5, \; -0.5, \; 0.0} \quad \overset{\text{element 2}}{0.5, \; -0.5, \; 0.0}]^T$$

$$\underline{\sigma}^1 = \boldsymbol{Q}^{-1}\boldsymbol{F}^1/t = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ -0.5 \\ 0.0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

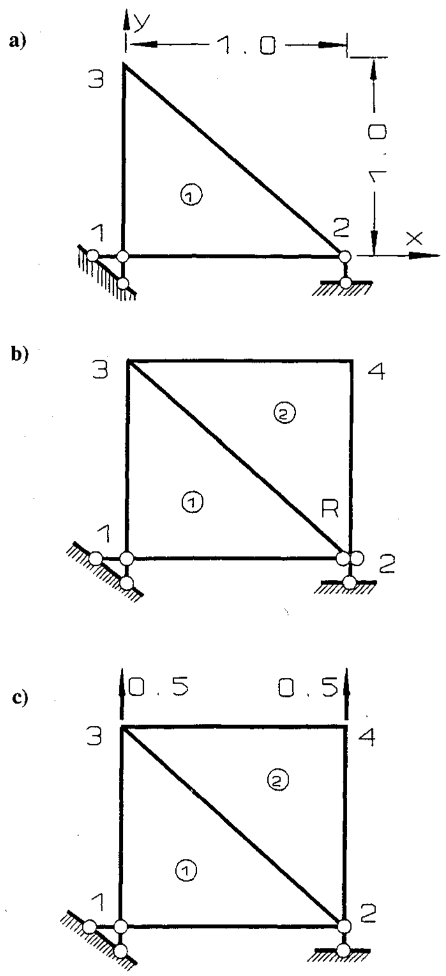


**Fig. 6 Structural variation process of a finite element system.**

$$\underline{\sigma}^2 = Q^{-1}F^2/t = [0,\ 1,\ 0]^T$$

$$D = V^T W^{-1} F = [-0.3,\ 0.0 \quad 0.0,\ 1.0 \quad -0.3,\ 1.0]$$

node 2 node 3 node 4

### B. Remarks on SVM

The preceding simple but detailed example has been given only for validation of the theory of structural variations. This paper does not suggest using SVM for the analysis of an unchanging structure under unchanging load (simple analysis), because the matrix $V$ constructed by SVM is actually the influence function (or Green function in mathematical words) of all of the internal forces of the structure (as Theorem 1 implies); so $V$ gives much more than is required by the simple analysis and hence needs more space for storage and more effort for computation. It is not suitable to compare SVM to the conventional displacement method based on simple analysis because they have different capabilities. However, this theory does supply a mathematical foundation for deriving new approaches to handle some more complicated problems more efficiently than the conventional displacement method, depending on the specific engineering areas under consideration, such as design sensitivities, eigenpair and their sensitivities, plastic-elastic analysis, structural reliability analysis, contact problems, propagation of cracks, etc., but not everywhere. Each derivation of such approaches needs additional theoretical work to apply this theory to the specific engineering area, leading to a separate paper volume, which is hard to be treated as an example in this paper. Actually, some of those applications have already been presented, e.g., Refs. 2, 3, 5 and 6, where the efficiency and the advantage of SVM can be found.

## V. General Procedure of Generating Subelements

The theorems and formulations presented in the preceding sections are valid generally for finite element systems, provided that the element stiffness matrix $K^\alpha$ is expressed in the form of

$$K^\alpha = \sum_{s=1}^{N} K_s^\alpha$$

[Eq. (17) with the definition (18)]. However, the subelements generated from different element models will have different features ($E_s^\alpha$ and $W_s^\alpha$). Next is given a general procedure of generating the subelements from any element models whose element stiffness matrix can be written as $K^\alpha = \int_\Omega B^T M B \, \mathrm{d}\Omega$ in the finite element theory, where $\Omega$ is the element volume and the elastic matrix $M$ is supposed to be symmetric and positive definite but may be anisotropic. By using the dimensionless local coordinates $\xi$, $\eta$, $\zeta$ (see, e.g., Ref. 4) $K^\alpha$ may be expressed as

$$K^\alpha = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B^T M B \det(J)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta \tag{59}$$

where $\det(J)$ is the determinant of Jacobian matrix $J$. The matrix $K^\alpha$ can also be evaluated by Gaussian quadrature with $N$ points in the sum of several constant matrices:

$$K^\alpha = \sum_{m=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} H_i H_j H_m [B^T M B \det(J)]|_{\xi_i \eta_j \zeta_m} \tag{60}$$

where $H_i$, $H_j$, and $H_m$ are the Gaussian weight coefficients, and $(\ )|_{\xi_i\eta_j\zeta_m}$ represents the value at the Gaussian point ($\xi_i$, $\eta_j$, $\zeta_m$). Equation (60) supplies exact $K^\alpha$ for simple integrands, whereas for complicated ones only an approximation. For simplicity, let $k \equiv \phi B^T M B$ represent the general term of the constant matrices (evaluated at a Gaussian point) in Eq. (60) without confusion, where $\phi$ stands for the scalar factor. Since $M$ is symmetric and positive definite, there exists an orthogonal matrix $Q$ of the same dimension as $M$, making $Q^T M Q$ diagonal (see, e.g., Ref. 7). Therefore, one can obtain the general form of Eq. (11) as

$$k = \phi B^T M B = (B^T Q^{-T})(\phi Q^T M Q)(Q^{-1} B) = H W H^T \tag{61}$$

$$H \equiv B^T Q^{-T} \equiv [E_1, E_2, \ldots, E_q] \tag{62}$$

$$W \equiv \phi Q^T M Q \equiv \mathrm{diag}(W_1, W_2, \ldots, W_q) \tag{63}$$

where $q$ is the rank of $M$. Thus, the subelement stiffness matrix $k_s$ is defined as

$$k_s \equiv W_s E_s (E_s)^T, \qquad s = 1, 2, \ldots, q \tag{64}$$

then,

$$k = \sum_{s=1}^{q} k_s \tag{65}$$

where $W_s$ is a diagonal element of $W$, serving as the subelement stiffness modulus, and $E_s$ is a column vector of $H$, serving as the subelement vector. However, the expression (64) is only one term in Eq. (60), corresponding to one Gaussian point. Returning to the notation used at the beginning and denoting this general term by $(k_s^\alpha)_{ijm}$, one has the subelement form of $K^\alpha$ for a finite element model in general:

$$K^\alpha = \sum_{m=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{s=1}^{q} (k_s^\alpha)_{ijm} \tag{66}$$

Then, the subsequent procedures of proving the related theorems and formulations will remain the same as those done in the preceding sections for the triangular element systems and in Ref. 1 for skeletal structures. From the preceding arguments, one can see that the matrix $Q$ is the key to the question. Nevertheless, since $M$ is at most of a dimension of 6, it is assumably not difficult to find out $Q$ explicitly. For instance, the matrix $M$ for an isotropic, homogenous solid in three dimensions is

$$M = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 1 & \gamma & 0 & 0 & 0 \\ \gamma & \gamma & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix} \tag{67}$$

where $\gamma = \nu/(1-\nu)$, $\delta = (1-2\nu)/[(2(1-\nu)]$, and the corresponding matrix $Q$ may be taken (and/or multiplied by a constant factor) as

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} & 0 & 0 & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} & 0 & 0 & 0 \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{68}$$

## VI. Concluding Remarks

1) This paper brings to light some useful properties of finite element systems and suggests a new tool of research, treating them from a new point of view. 2) The new tool, SVM, has distinct features: it eliminates the need of assembling and solving simultaneous equations. This is favorable for the analysis of those structures where structural modifications and design sensitivities are required. For instance, to change a complete system into a discontinuous one with a crack, what one has to do is only the removal of a constraint-subelement; this can be done easily by using Eq. (55). 3) SVM is inherently suitable for parallel computations. The BD matrix $V$ of a

system can be built up by parts separately and parallelly first and then integrate them by using constraint-subelements for the final $\boldsymbol{V}$. 4) SVM is favorable for interface design on computers because any local modification of a structure can be easily done by SVM, consuming much less time than by the commonly used FEM. 5) The present theory also applies to nonlinear materials.

## Appendix

This Appendix gives short proofs of the five theorems used in Secs. III.A–III.D (which are given in Ref. 1, but in Chinese). The notations already defined in the preceding sections will not be restated herein.

*Theorem 1:* Suppose that a $\bar{\boldsymbol{P}}_r^\ell$ is applied at $\binom{\ell}{r}$; then the corresponding displacement vector, denoted by $\bar{\boldsymbol{D}}_r^\ell$, is determined by $\bar{\boldsymbol{D}}_r^\ell = \boldsymbol{K}^{-1}\bar{\boldsymbol{P}}_r^\ell$. According to Eqs. (22), (21), and (25), one has the conclusion of Eq. (28):

$$\bar{F}_{sr}^{\alpha\ell} = W_s^\alpha (\boldsymbol{E}_s^\alpha)^T \bar{\boldsymbol{D}}_r^\ell = (\boldsymbol{P}_s^\alpha)^T \boldsymbol{K}^{-1} \bar{\boldsymbol{P}}_r^\ell = (\boldsymbol{K}^{-1}\boldsymbol{P}_s^\alpha)^T \bar{\boldsymbol{P}}_r^\ell$$

$$= (\boldsymbol{V}_s^\alpha)^T \bar{\boldsymbol{P}}_r^\ell = V_{sr}^{\alpha\ell} \tag{A1}$$

*Theorem 2:* Since $\boldsymbol{K}^\alpha = \boldsymbol{H}^\alpha \boldsymbol{W}^\alpha (\boldsymbol{H}^\alpha)^T$ [Eq. (11)], one can rewrite

$$\boldsymbol{K} = \sum_{\alpha=1}^{m} \boldsymbol{K}^\alpha$$

as

$$\boldsymbol{K} = \boldsymbol{H}\boldsymbol{W}\boldsymbol{H}^T \tag{A2}$$

where $\boldsymbol{H}^\alpha$ may involve constraint- or support-subelements and

$$\boldsymbol{H} = [\boldsymbol{H}^1, \boldsymbol{H}^2, \ldots, \boldsymbol{H}^m] \tag{A3}$$

Then one has the conclusion of Theorem 2:

$$\boldsymbol{K}^{-1} = \boldsymbol{K}^{-1}\boldsymbol{K}\boldsymbol{K}^{-1} = \boldsymbol{K}^{-1}\boldsymbol{H}\boldsymbol{W}\boldsymbol{H}^T\boldsymbol{K}^{-1} = \boldsymbol{K}^{-1}\boldsymbol{H}\boldsymbol{W}\boldsymbol{W}^{-1}\boldsymbol{W}\boldsymbol{H}^T\boldsymbol{K}^{-1}$$

$$= (\boldsymbol{K}^{-1}\boldsymbol{H}\boldsymbol{W})\boldsymbol{W}^{-1}(\boldsymbol{K}^{-1}\boldsymbol{H}\boldsymbol{W})^T = \boldsymbol{V}^T\boldsymbol{W}^{-1}\boldsymbol{V}$$

*Theorem 3:* According to the definitions (26), (25), and (21), one has the following conclusion:

$$Z_{sr}^{\alpha\beta} = (\boldsymbol{E}_s^\alpha)^T \boldsymbol{V}_r^\beta = (\boldsymbol{E}_s^\alpha)^T \boldsymbol{K}^{-1} \boldsymbol{E}_r^\beta W_r^\beta = W_s^\alpha (\boldsymbol{E}_s^\alpha)^T \boldsymbol{K}^{-1} \boldsymbol{E}_r^\beta W_r^\beta / W_s^\alpha$$

$$= (\boldsymbol{K}^{-1}\boldsymbol{P}_s^\alpha)^T \boldsymbol{E}_r^\beta W_r^\beta / W_s^\alpha = (\boldsymbol{V}_s^\alpha)^T \boldsymbol{E}_r^\beta W_r^\beta / W_s^\alpha = Z_{rs}^{\beta\alpha} W_r^\beta / W_s^\alpha$$

*Theorem 4:* Suppose $W_s^\alpha$ is changed into $\hat{W}_s = W_s^\alpha + \Delta W_s^\alpha$ where $\Delta W_s^\alpha$ stands for any increment of $W_s^\alpha$; then from the definition of the BD vector, the new one, $\hat{\boldsymbol{V}}_s^\alpha$, must satisfy $(\boldsymbol{K} + \Delta\boldsymbol{K})\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{P}_s^\alpha + \Delta\boldsymbol{P}_s^\alpha = \boldsymbol{E}_s^\alpha(W_s^\alpha + \Delta W_s^\alpha) = \boldsymbol{P}_s^\alpha(1 + m_s^\alpha)$. However, due to the variation of a single $W_s^\alpha$, one has $\Delta\boldsymbol{K} = \Delta\boldsymbol{K}^\alpha = \boldsymbol{E}_s^\alpha(\boldsymbol{E}_s^\alpha)^T \Delta W_s^\alpha = \boldsymbol{P}_s^\alpha m_s^\alpha (\boldsymbol{E}_s^\alpha)^T$; therefore, $[\boldsymbol{K} + \boldsymbol{P}_s^\alpha m_s^\alpha (\boldsymbol{E}_s^\alpha)^T]\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{P}_s^\alpha(1 + m_s^\alpha)$. Premultiplying the last equation by $\boldsymbol{K}^{-1}$ yields

$$\hat{\boldsymbol{V}}_s^\alpha = -\boldsymbol{K}^{-1}\boldsymbol{P}_s^\alpha m_s^\alpha (\boldsymbol{E}_s^\alpha)^T \hat{\boldsymbol{V}}_s^\alpha + \boldsymbol{K}^{-1}\boldsymbol{P}_s^\alpha(1 + m_s^\alpha) = -\boldsymbol{V}_s^\alpha m_s^\alpha \hat{Z}_{ss}^{\alpha\alpha}$$

$$+ \boldsymbol{V}_s^\alpha(1 + m_s^\alpha) = \boldsymbol{V}_s^\alpha(1 + m_s^\alpha - m_s^\alpha \hat{Z}_{ss}^{\alpha\alpha}) \tag{A4}$$

Premultiplying the preceding equation by $(\boldsymbol{E}_s^\alpha)^T$ yields $\hat{Z}_{ss}^{\alpha\alpha} = Z_{ss}^{\alpha\alpha}(1 + m_s^\alpha - m_s^\alpha \hat{Z}_{ss}^{\alpha\alpha})$ from which one has

$$\hat{Z}_{ss}^{\alpha\alpha} = Z_{ss}^{\alpha\alpha}/(1 + m_s^\alpha Z_{ss}^{\alpha\alpha}) \tag{A5}$$

Substituting Eq. (A5) into Eq. (A4) yields Eq. (35), and repeating the same procedure and noting $\Delta\boldsymbol{P}_r^\beta = \boldsymbol{0}$ when $W_s^\alpha$ varies will give Eq. (36).

*Proof of Eqs. (53) and (54) (Part of Theorem 5):* Let the new connecting subelement $\binom{\alpha}{s}$ have its $W_s^\alpha$, $\boldsymbol{E}_s^\alpha$, and $\boldsymbol{P}_s^\alpha = W_s^\alpha \boldsymbol{E}_s^\alpha$; then through a procedure similar to that done for Eq. (A4), one has $\hat{\boldsymbol{K}}\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{P}_s^\alpha$ or $[\boldsymbol{K} + \boldsymbol{P}_s^\alpha(\boldsymbol{E}_s^\alpha)^T]\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{P}_s^\alpha$ or $\boldsymbol{K}\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{P}_s^\alpha(1 - \hat{Z}_{ss}^{\alpha\alpha})$. Therefore, one has

$$\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{K}^{-1}\boldsymbol{P}_s^\alpha(1 - \hat{Z}_{ss}^{\alpha\alpha}) = (1 - \hat{Z}_{ss}^{\alpha\alpha})\mathring{\boldsymbol{V}}_s^\alpha \tag{A6}$$

where $\mathring{\boldsymbol{V}}_s^\alpha$ is the auxiliary basic displacement vector that can be obtained directly from Theorem 2:

$$\mathring{\boldsymbol{V}}_s^\alpha \equiv \boldsymbol{K}^{-1}\boldsymbol{P}_s^\alpha = \boldsymbol{V}^T\boldsymbol{W}^{-1}\boldsymbol{V}\boldsymbol{P}_s^\alpha \tag{A7}$$

Premultiplying Eq. (A6) by $(\boldsymbol{E}_s^\alpha)^T$ yields

$$\hat{Z}_{ss}^{\alpha\alpha} = \mathring{Z}_{ss}^{\alpha\alpha}/(1 + \mathring{Z}_{ss}^{\alpha\alpha}) \tag{A8}$$

where

$$\mathring{Z}_{ss}^{\alpha\alpha} \equiv (\boldsymbol{E}_s^\alpha)^T \mathring{\boldsymbol{V}}_s^\alpha \tag{A9}$$

Substituting Eq. (A8) into Eq. (A6) yields Eq. (53), i.e.,

$$\hat{\boldsymbol{V}}_s^\alpha = \mathring{\boldsymbol{V}}_s^\alpha/(1 + \mathring{Z}_{ss}^{\alpha\alpha}) \tag{A10}$$

and going through the similar procedure gives Eq. (54).

*Proof of Eq. (52) (Another Part of Theorem 5):* A constraint-subelement or support-subelement $\binom{R}{t}$ is a special case of a connecting beam-subelement with $W_t^R = \infty$ (Fig. 5). Actually, one can treat it as $W_t^R \to \infty$. So, before it becomes $\infty$, Eq. (54) can apply to the case of adding $\binom{R}{t}$ with $W_t^R < \infty$. Thus, one has

$$\hat{\boldsymbol{V}}_r^\beta = \{\boldsymbol{V}_r^\beta - Z_{tr}^{R\beta}(\mathring{\boldsymbol{V}}_t^R)^*/[1 + (\mathring{Z}_{tt}^{RR})^*]\}|_{W_t^R \to \infty} \tag{A11}$$

where $(\mathring{\boldsymbol{V}}_t^R)^* \equiv \boldsymbol{K}^{-1}\boldsymbol{E}_t^R W_t^R = \mathring{\boldsymbol{V}}_t^R W_t^R$; $\mathring{\boldsymbol{V}}_t^R \equiv \boldsymbol{K}^{-1}\boldsymbol{E}_t^R$; $(\mathring{Z}_{tt}^{RR})^* \equiv (\boldsymbol{E}_t^R)^T(\mathring{\boldsymbol{V}}_t^R)^* = (\boldsymbol{E}_t^R)^T\mathring{\boldsymbol{V}}_t^R W_t^R = \mathring{Z}_{tt}^{RR} W_t^R$. Thus, one has

$$(\mathring{\boldsymbol{V}}_t^R)^*/[1 + (\mathring{Z}_{tt}^{RR})^*] = \mathring{\boldsymbol{V}}_t^R W_t^R/(1 + \mathring{Z}_{tt}^{RR} W_t^R)$$
$$\{(\mathring{\boldsymbol{V}}_t^R)^*/[1 + (\mathring{Z}_{tt}^{RR})^*]\}|_{W_t^R \to \infty} = \mathring{\boldsymbol{V}}_t^R/\mathring{Z}_{tt}^{RR} \tag{A12}$$

Substituting Eq. (A12) back into Eq. (A11) yields Eq. (52).

*Proof of Eq. (55):* Again, let the support subelement $\binom{R}{t}$ (Fig. 5) be treated as $W_t^R \to \infty$. Then, before $W_t^R$ becomes $\infty$, Eq. (36) can apply to the removal of $\binom{R}{t}$ by setting $m_t^R = -1$. Therefore,

$$\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{V}_s^\alpha + \boldsymbol{V}_t^R Z_{ts}^{R\alpha}/(1 - Z_{tt}^{RR}) \tag{A13}$$

Using Theorem 3 to substitute $Z_{st}^{\alpha R} W_s^\alpha / W_t^R$ for $Z_{ts}^{R\alpha}$ in Eq. (A13), one has

$$\hat{\boldsymbol{V}}_s^\alpha = \boldsymbol{V}_s^\alpha + \boldsymbol{V}_t^R W_s^\alpha Z_{st}^{\alpha R} D_t^R \tag{A14}$$

where

$$D_t^R \equiv 1/[(1 - Z_{tt}^{RR})W_t^R] \tag{A15}$$

or in the component form for any DOF $\binom{\ell}{r}$,

$$\hat{V}_{sr}^{\alpha\ell} = V_{sr}^{\alpha\ell} + V_{tr}^{R\ell} W_s^\alpha Z_{st}^{\alpha R} D_t^R \tag{A16}$$

or

$$\hat{\boldsymbol{V}}_{\bullet r}^{\alpha\ell} = \boldsymbol{V}_{\bullet r}^{\alpha\ell} + V_{tr}^{R\ell} \boldsymbol{W}^\alpha \boldsymbol{Z}_{\bullet t}^{\alpha R} D_t^R \tag{A17}$$

where $\hat{\boldsymbol{V}}_{\bullet r}^{\alpha\ell}$, $\boldsymbol{V}_{\bullet r}^{\alpha\ell}$, and $\boldsymbol{Z}_{\bullet}^{\alpha R}$ have been defined in Secs. III.A and III.D.

Suppose there are in total $q$ elements numbered $\beta = 1, 2, \ldots, q$ around the node $R$ where the support-subelement $\binom{R}{t}$ is to be removed (Fig. 5). From Eq. (38), one has the nodal force vector $\boldsymbol{f}^\beta$ expressed as

$$\boldsymbol{f}^\beta = \boldsymbol{H}^\beta \boldsymbol{F}^\beta \tag{A18}$$

Then the nodal force vector $f^{\beta}$ due to a unit load vector $\bar{P}_r^{\ell}$ applied at any DOF $\binom{\ell}{r}$ should be expressed by Eq. (A18), using the notations defined in Sec. II.C, as

$$\bar{f}^{\beta} = H^{\beta}\bar{F}_{\bullet r}^{\beta\ell} \tag{A19}$$

Thus the force vector at node $R$ (part of $\bar{f}^{\beta}$), denoted by $\bar{f}_R^{\beta}$, is

$$\bar{f}_R^{\beta} = H_R^{\beta}\bar{F}_{\bullet r}^{\beta\ell} \tag{A20}$$

Projecting $\bar{f}_R^{\beta}$ onto $\binom{R}{t}$ by using $R_t^R$ and $T_t^{\beta}$ (see Sec. III.D) gives the force component in this direction:

$$(R_t^R)^T\bar{f}_R^{\beta} = (R_t^R)^T H_R^{\beta}\bar{F}_{\bullet r}^{\beta\ell} = -(T_t^{\beta})^T\bar{F}_{\bullet r}^{\beta\ell} \tag{A21}$$

Nevertheless, according to Theorem 1, Eq. (A21) can be rewritten as

$$(R_t^R)^T\bar{f}_R^{\beta} = -(T_t^{\beta})^T V_{\bullet r}^{\beta\ell} \tag{A22}$$

Equation (A22) is a general expression for the nodal force component associated with element β in a given direction $\binom{R}{t}$, valid for the node $R$ either with the support subelement $\binom{R}{t}$ or without it. Applying Eq. (A22) to the node $R$ after removing $\binom{R}{t}$, then the total of these components from all elements connected to it must be balanced, i.e.,

$$\sum_{\beta=1}^{q}(R_t^R)^T\bar{f}_R^{\beta} = -\sum_{\beta=1}^{q}(T_t^{\beta})^T\hat{V}_{\bullet r}^{\beta\ell} = 0 \tag{A23}$$

and using Eq. (A17), one has

$$\left[-\sum_{\beta=1}^{q}(T_t^{\beta})^T V_{\bullet r}^{\beta\ell}\right] - \left[\sum_{\beta=1}^{q}(T_t^{\beta})^T W^{\beta}Z_{\bullet r}^{\beta\ell}\right]V_{tr}^{R\ell}D_t^R = 0 \tag{A24}$$

Nevertheless, according to Eq. (A22), the first part of Eq. (A24) is the total force component from all of the elements around $R$ before removing the support subelement $\binom{R}{t}$; it should be balanced with the basic internal force of $\binom{R}{t}$. Therefore, from Theorem 1, it must be equal to $V_{tr}^{R\ell}$, i.e.,

$$-\sum_{\beta=1}^{q}(T_t^{\beta})^T V_{\bullet r}^{\beta\ell} = V_{tr}^{R\ell} \tag{A25}$$

Thus, from Eqs. (A25) and (A24) one has

$$\left[1 - \sum_{\beta=1}^{q}(T_t^{\beta})^T W^{\beta}Z_{\bullet r}^{\beta\ell}D_t^R\right]V_{tr}^{R\ell} = 0$$

from which comes

$$D_t^R = 1/\left[\sum_{\beta=1}^{q}(T_t^{\beta})^T W^{\beta}Z_{\bullet r}^{\beta\ell}\right] \tag{A26}$$

Substituting Eq. (A26) back into Eq. (A14) and letting $W_t^R \to \infty$, one has the conclusion of Eq. (55):

$$\hat{V}_s^{\alpha} = V_s^{\alpha} + V_t^R W_s^{\alpha}Z_{st}^{\alpha R}/\left[\sum_{\beta=1}^{q}(T_t^{\beta})^T W^{\beta}Z_{\bullet r}^{\beta\ell}\right] = V_s^{\alpha} + V_t^R\eta_{ts}^{R\alpha} \tag{A27}$$

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